# A Gentle and Incomplete Introduction to Branch and Cut with Intersection Cuts

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SIAM Student Chapter

# Agenda

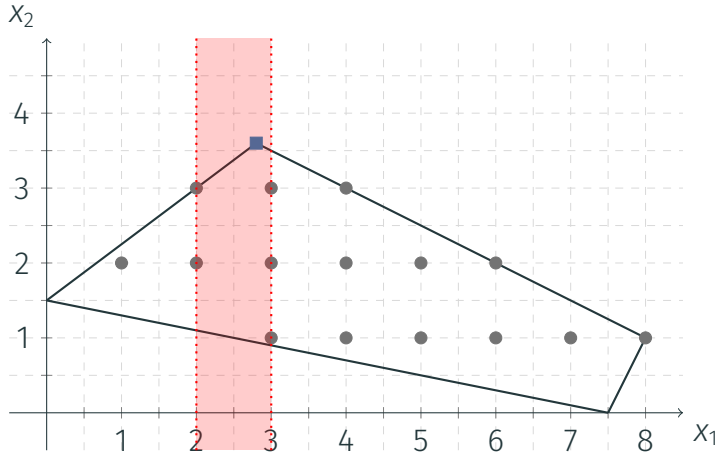
- Mixed-Integer Linear Optimization
  - Branch and Bound for MILPs
  - Branch and Cut for MILPs
  - Intersection Cuts for MILPs
- Mixed-Integer Linear Bilevel Optimization
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# Mixed-Integer Linear Optimization

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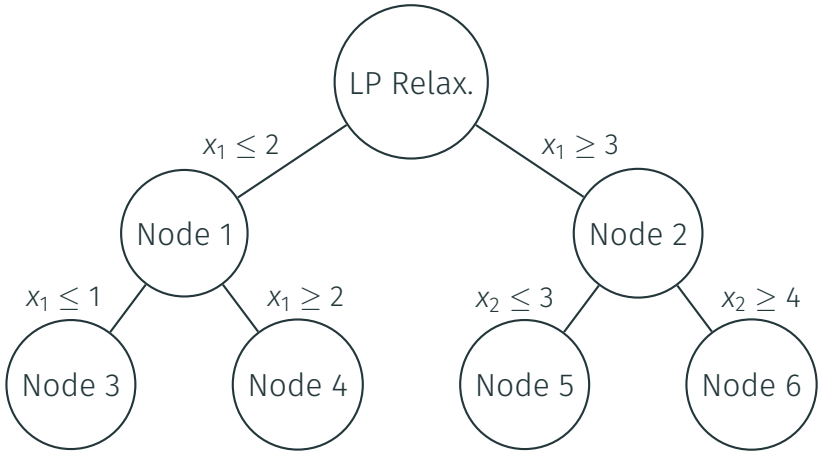

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# Example




TikZ images adapted from Beck et al. 2025

# Example


LP Relax.
$x_1 \leq 2$
$x_1 \geq 3$
Node 1
Node 2
$x_1 \leq 1$
$x_1 \geq 2$
$x_2 \leq 3$
$x_2 \geq 4$
Node 3
Node 4
Node 5
Node 6

# Pruning Rules

Fathoming by . . .

- **Integrality**: Node's LP relaxation yields an integer solution
  - This solution may improve the current best known solution
- **Infeasibility**: Node's LP relaxation is infeasible
- **Bound**: Node solution is worse than the current best known solution

Integer infeasible solution of node's LP relaxation $\implies$ (local) lower bound

Integer feasible solution of node's LP relaxation $\implies$ (global) upper bound

After searching the tree, we will have found an optimal solution or proved that no feasible solution exists.

# Advantages of Branch and Bound

+ Guaranteed to find an optimal solution (or prove infeasibility)
+ Can be applied to nearly any MIP if you can
  - Define a relaxation
  - Partition the search space
+ Even if stopped early, it gives a certificate of how far the best known solution is from optimality

# Disadvantages of Branch and Bound

— In the worst case, the size of the B&B tree grows exponentially with the number of variables

$\Rightarrow$ Large computation times

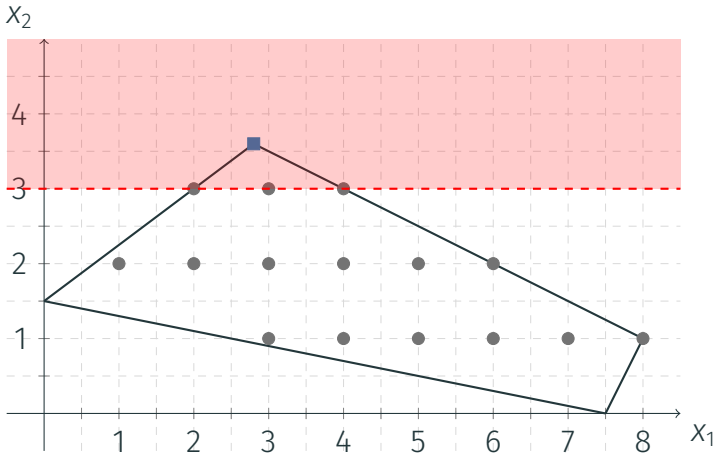
$\Rightarrow$ Excessive memory usage

- **Can We Do Better?**
  - Yes! By adding additional **cuts** to the LP relaxations at each node
  - **Branch and Cut (and Bound)!**

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# Example



# Globally Valid Inequality

We call an inequality

$$\alpha^\top x \geq \gamma$$

with $\alpha \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$ **globally valid** for the MILP feasible set $X^{\text{MILP}}$ if

$$X^{\text{MILP}} \subseteq \left\{x \in X^{\text{MILP}} \colon \alpha^\top x \geq \gamma\right\}$$

holds.

# Locally Valid Inequality

We call an inequality

$$\alpha^\top x \geq \gamma$$

with $\alpha \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$ **locally valid** for the restricted feasible set $X_k^{\text{MILP}} = X^{\text{MILP}} \cap \bar{\Omega}_k$ at node $k$ of a branch-and-cut search tree if

$$X_k^{\text{MILP}} \subseteq \left\{x \in X_k^{\text{MILP}} \colon \alpha^\top x \geq \gamma\right\}$$

holds.

# Cutting Plane

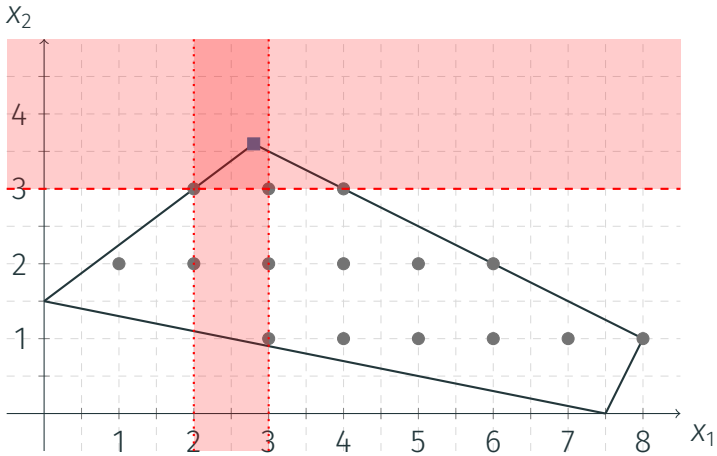
A **cutting plane** (or simply **cut**) for a MILP feasible set $X^{\text{MILP}}$ is a (globally) valid inequality

$$\alpha^\top x \geq \gamma$$

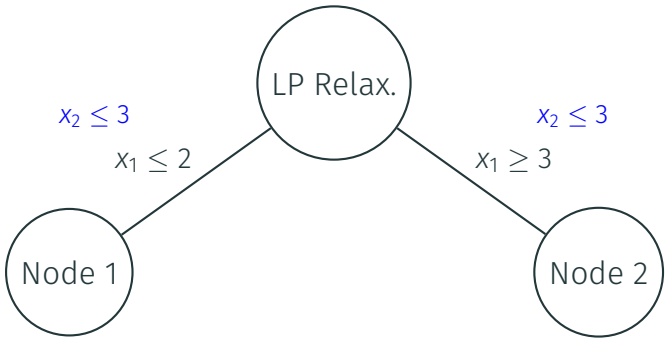
that removes at least one infeasible point from the LP relaxation $X^{\text{LP}}$, i.e., there exists $\bar{x} \in X^{\text{LP}} \setminus X^{\text{MILP}}$ such that

$$\alpha^\top \bar{x} < \gamma.$$

# Example



# Example



# Advantages of Branch and Cut

+ Tighter LP relaxations at each node
+ Fewer nodes need to be explored

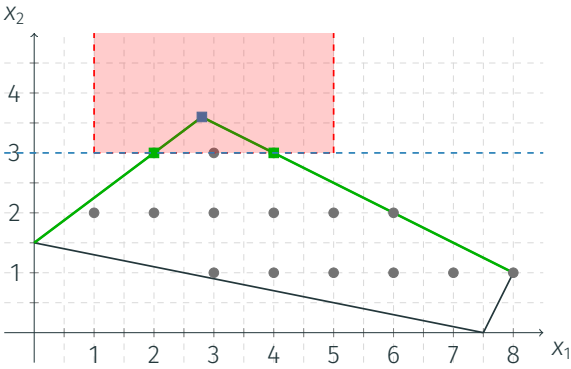
$\Rightarrow$ Reduced computation times

$\Rightarrow$ Lower memory usage

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  - Intersection Cuts for MIBLPs

# Intersection Cuts for MILPs




- Lattice free set L
  - Contains infeasible point $\bar{x}$ in its interior
  - Contains no feasible point in its interior
- Extreme rays of the LP relaxation
- Intersection points of extreme rays with boundary of L
- **Cut:** Hyperplane through intersection points

# Mixed-Integer Linear Bilevel Optimization

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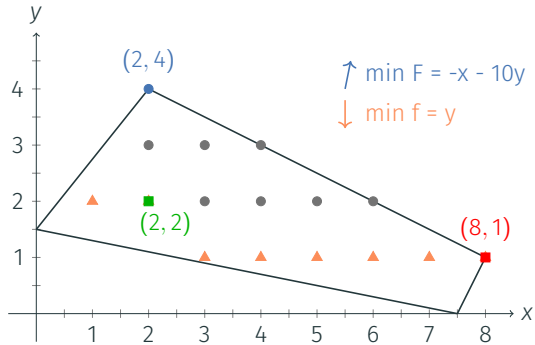
# Mixed-Integer Bilevel Linear Programs (MIBLPs)

$$
\begin{aligned}
\min_{x,y} \quad & F(x,y) = c_x^\top x + c_y^\top y \\
\text{s.t.} \quad & Ax + By \geq a \\
& x_i \in \mathbb{Z}, \quad i \in I_x \\
& y \in \mathcal{R}(x)
\end{aligned}
$$

where $\mathcal{R}(x)$ is the solution set of the lower-level problem

$$
\begin{aligned}
\min_{y} \quad & f(x,y) = d^\top y \\
& Cx + Dy \geq b \\
& l \leq y \leq u \\
& y_i \in \mathbb{Z}, \quad i \in I_y
\end{aligned}
$$

# Branch and Bound for MIBLPs




$$F(2,2) = -22, \quad F(8,1) = -18, \quad F(2,4) = -42$$

$\Longrightarrow$ Continuous counterpart to the bilevel problem is not a relaxation!

$\Longrightarrow$ Continuous relaxation of the single-level reformulation is a relaxation!

# Pruning Rules

Fathoming by . . .

- ~~**Integrality**: Node's BLP relaxation yields an integer solution~~
- **Infeasibility**: Node's BLP relaxation is infeasible
- **Bound**: Node solution is worse than the current best known solution

The **bounding** rule can be used with the linearized single-level reformulation (see Beck et al. 2025)

# When we find a node solution with better value than incumbent

- ... we need to check for integer feasibility

  $\Longrightarrow$ Easy!

- ... we need to check for bilevel feasibility
  - Solve the lower-level problem with fixed upper-level variables
  - NP-hard mixed-integer problem

  $\Longrightarrow$ Not so easy!

$\Longrightarrow$ Branch and Bound for Bilevel Problems is much more difficult!

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# Globally Valid Inequalities for MIBLPs

We call an inequality

$$\alpha^\top x + \beta^\top y \geq \gamma$$

with $\alpha \in \mathbb{R}^{n_x}$, $\beta \in \mathbb{R}^{n_y}$, and $\gamma \in \mathbb{R}$ **globally valid** for the feasible set $\mathcal{F}$ of the bilevel problem if

$$\mathcal{F} \subseteq \left\{(x, y) \in X \times Y \colon \alpha^\top x + \beta^\top y \geq \gamma\right\}$$

holds.

# Locally Valid Inequalities for MIBLPs

We call an inequality

$$\alpha^\top x + \beta^\top y \geq \gamma$$

with $\alpha \in \mathbb{R}^{n_x}$, $\beta \in \mathbb{R}^{n_y}$, and $\gamma \in \mathbb{R}$ **locally valid** for the restricted feasible set $\mathcal{F}_k = \mathcal{F} \cap \bar{\Omega}_k$ at node $k$ of a branch-and-cut search tree if

$$\mathcal{F}_k \subseteq \left\{(x, y) \in X \times Y \colon \alpha^\top x + \beta^\top y \geq \gamma\right\}$$

holds.

# Separation Problem

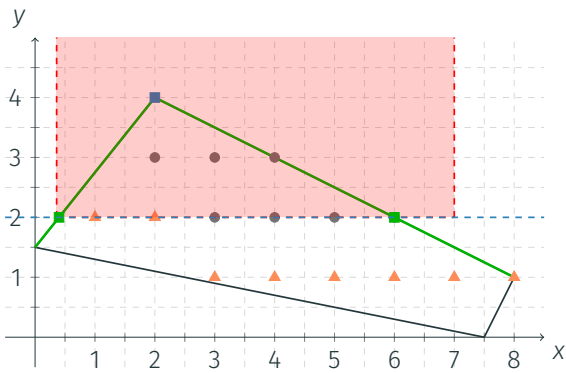
The **separation problem** for $\mathcal{F}$ is to determine whether a given point $(\bar{x}, \bar{y}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ is feasible for the bilevel problem and, if not, to generate $\alpha \in \mathbb{R}^{n_x}$, $\beta \in \mathbb{R}^{n_y}$, and $\gamma \in \mathbb{R}$ such that

$$\alpha^\top x + \beta^\top y \geq \gamma \quad \text{for all } (x, y) \in \mathcal{F} \quad \text{and} \quad \alpha^\top \bar{x} + \beta^\top \bar{y} < \gamma.$$

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# Example




- Bilevel free set S
  - Contains bilevel infeasible point $(\bar{x}, \bar{y})$ in its interior
  - Contains no bilevel feasible point in its interior
- Extreme rays of the LP relaxation
- Intersection points of extreme rays with boundary of S
- **Cut:** Hyperplane through intersection points

# Intersection Cuts for MIBLPs

- — Computational performance highly dependent on choice of bilevel free set
- — Numerically unstable in some cases
- — Difficult to implement using known solvers
- — May become quite shallow if applied to the same node multiple times
- — Only locally valid
- \+ Very generic and can be applied to most MIBLPs
- \+ Can be generated relatively efficiently if the bilevel free set is known

# Outlook

- Apply these techniques to decision-dependent robust optimization problems
- ”Project” the intersection cuts into the leader space for min-max problems