Is Two the Best We Can Do?

Vertex Covers and the Unique Games Conjecture

Simon Stevens

November 3, 2026

SIAM Student Chapter, Trier University

Overview

Vertex Cover

Can Anyone Beat 2?

The Unique Games Conjecture

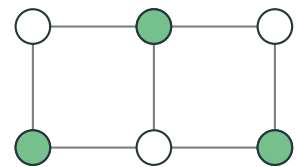
Vertex Cover

Vertex Cover: Guarding Streets

- A city is a graph $G = (V, E)$: corners and streets
- Place guards on corners so that **every street has a guard at one of its ends**
- Such a guard set is a **vertex cover**; $\tau(G)$ is the smallest number of guards that suffices
- **Vertex Cover**, computing $\tau(G)$, is NP-hard (Karp 1972)
- So we settle for fast methods with a guarantee:

Factor-α method
Always returns a valid guard set with at most $\alpha \cdot \tau(G)$ guards

- How small can α be?



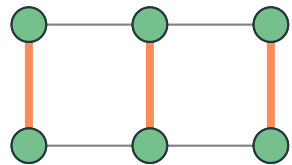
$\tau(G) = 3$: every street is watched

Guard Both Ends

- Pick any street without a guard and guard **both** its ends. Repeat until every street is watched
- **Valid:** we only stop when every street is watched
- **Fast:** each round watches at least one new street, so at most #streets rounds, each one pass over the streets: polynomial time
- **At most twice optimal:** the picked streets share no corner, so $\#\text{picked} \leq \tau(G)$ and

$$\#\text{guards} = 2 \cdot \#\text{picked} \leq 2\tau(G)$$

- **Factor 2 is reached:** picking the three vertical streets gives 6 guards, but 3 suffice

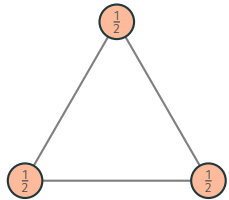


picked: 3 streets (orange)

guards: $6 = 2 \cdot 3 = 2\tau(G)$

Allow Half Guards

- Relax the rules: each corner v gets an amount $x_v \in [0, 1]$ of a guard
- A street uv is watched if $x_u + x_v \geq 1$
- Minimize the total amount $\sum_v x_v$
- Every real guard set is allowed, so the best fractional total is $\leq \tau(G)$
- The best fractional solution can be **computed fast**
- Triangle: three half guards cost $\frac{3}{2}$, real guards cost 2



total $\frac{3}{2} < 2 = \tau(G)$

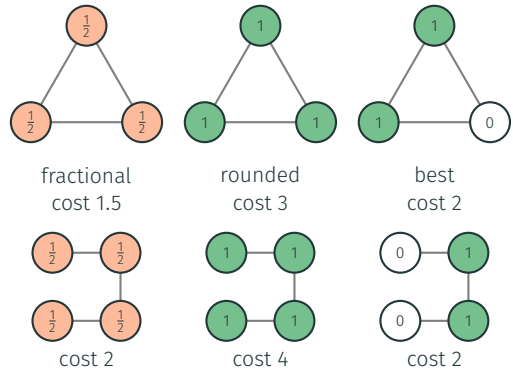
Round Up the Halves

- Take a best fractional solution x and put a real guard on every corner with $x_v \geq \frac{1}{2}$
- Some best solution uses only 0, $\frac{1}{2}$, 1 (Nemhauser and Trotter 1974): only the halves cost us
- **Valid:** $x_u + x_v \geq 1$ forces $x_u \geq \frac{1}{2}$ or $x_v \geq \frac{1}{2}$
- **At most twice optimal:**

$$\#\text{guards} \leq \sum_v 2x_v \leq 2\tau(G)$$

- A second factor-2 method: one fast computation, one rounding step
- **Factor 2 is reached:** on a path with 4 corners, all halves is a best fractional solution (cost 2), and rounding guards all 4 corners where 2 suffice

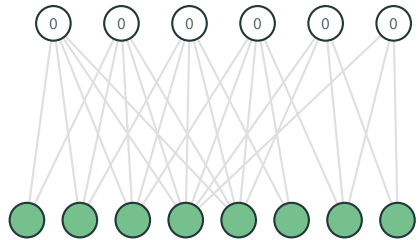
Can any fast method beat factor 2 in the worst case?



Can Anyone Beat 2?

Busiest Corner First?

- Guard the corner with the most unwatched streets; repeat
- Step 1: the lower corner with 6 beats every upper corner (5)
- Each step removes one street from many upper corners, so a lower corner stays the busiest
- Result: 8 guards.



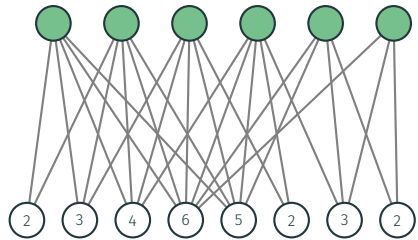
steps 6–8: guard the corners with 2: 8 **guards**

Busiest Corner First?

- Guard the corner with the most unwatched streets; repeat
- Step 1: the lower corner with 6 beats every upper corner (5)
- Each step removes one street from many upper corners, so a lower corner stays the busiest
- Result: 8 guards. But 6 suffice
- The same pattern with more upper corners:

upper corners	busiest first	ratio
6	8	1.33
24	60	2.50
100	382	3.82

- The ratio keeps growing (like ln of the size): **no fixed factor at all**



best: the 6 upper corners

What Was Known Before 2026

Fast methods for Vertex Cover (upper bounds)

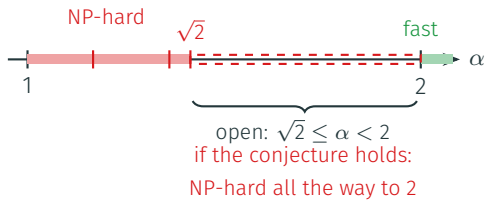
- Factor 2: guard both ends, or round up the halves
- Best known: $2 - \Theta\left(\frac{1}{\sqrt{\log n}}\right)$ (Karakostas 2009), still tends to 2

NP-hard to beat (lower bounds)

- $\frac{7}{6} \approx 1.17$ (Håstad 2001)
- 1.36 (Dinur and Safra 2005)
- $\sqrt{2} \approx 1.41$ (Khot et al. 2023)

If the Unique Games Conjecture holds

- beating 2 is NP-hard (Khot and Regev 2008)



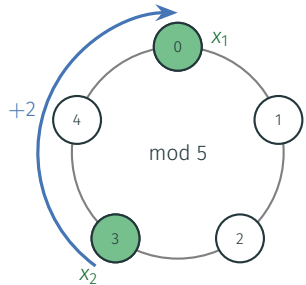
The Unique Games Conjecture

Equations on a Clock

- Values $0, 1, 2, 3, 4$ on a clock with 5 hours: after 4 comes 0 again (“modulo 5”)
- An equation like

$$x_1 - x_2 \equiv 2 \pmod 5$$

  says: x_1 **is two hours ahead of** x_2
- Example: $x_2 = 3$ gives $x_1 = 0$ $(3 \to 4 \to 0)$. Each value of one variable fixes **exactly one** value of the other: the equation is **unique**

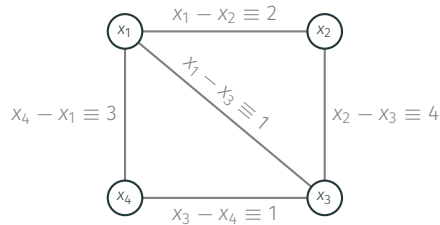


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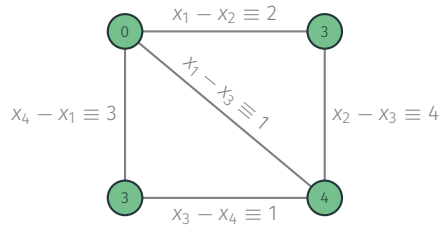
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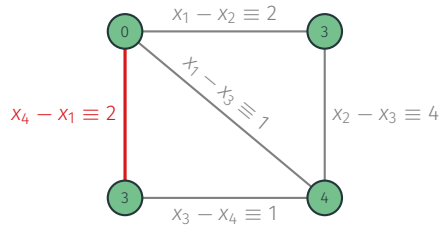
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- **The puzzle:** many variables (dots) and equations (lines); find values that satisfy as many equations as possible
- **All satisfiable?** Easy: set $x_1 = 0$ and follow the lines
- **One constant changed:** the cycle no longer fits, at most 4 of 5 hold. In a large puzzle where only 99 % can hold, following the lines breaks down

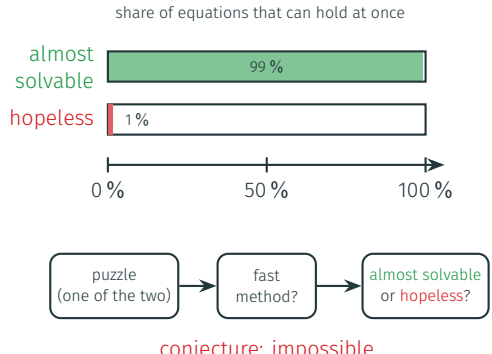


all modulo 5

no values satisfy all 5

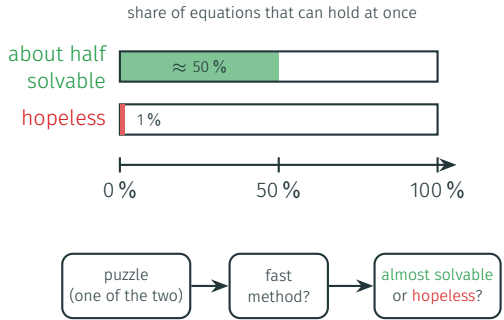
The Unique Games Conjecture (Khot 2002)

- You get a large puzzle with a **promise**: it is one of two kinds
  - almost solvable: 99 % of the equations can hold at once
  - hopeless: at most 1 % can hold at once
- **Conjecture:** no fast method can tell which kind it is
- Why is this hard? With 100 % we just follow the lines (previous slide). With 1 % broken, we do not know which lines to trust



The Unique Games Conjecture (Khot 2002)

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- Why is this hard? With 100 % we just follow the lines (previous slide). With 1 % broken, we do not know which lines to trust
- **Proved in 2018** (Khot et al. 2023): telling "about half solvable" from hopeless is **NP-hard**. Consequence: Vertex Cover with factor below $\sqrt{2}$ is NP-hard



for "about half": proved NP-hard (2018)

The Claim (October 2026)

- **Claim 1** (OpenAI 2026b): the Unique Games Conjecture is true
- With Khot and Regev (2008): **no fast method beats factor** 2 **for Vertex Cover**; the simple rounding is optimal
- **Claim 2** (OpenAI 2026a): a **direct** proof of the same, without the conjecture. Both with Lean formalizations, neither refereed

If Claim 1 holds, simple methods are optimal elsewhere too

- Max-Cut: $\alpha \approx 0.878$ (Khot et al. 2007)
- Max Acyclic Subgraph: $\alpha = \frac{1}{2}$ (Guruswami et al. 2008)
- Graph Pricing: $\alpha = \frac{1}{4}$ (Lee 2014)
- Every Max-CSP: α given by its basic SDP relaxation (Raghavendra 2008)

Good news for twenty years of results “assuming the conjecture”.
Bad news for everyone hunting for better algorithms.

But if you want to win 1 Million Dollars: find a fast guard method with factor $2 - \varepsilon$! ;)